The time period of flat spiral spring :—

Let us suppose that the load W is attached to the free end of the flat spiral spring. This load produced negligibly small bending and depression due to twisting of the wire.

A shearing force W acts vertically over the lower section A, together with the twisting couple produces a twist $\phi$ in the wire and is balanced by the restoring couple $\frac{\pi \eta r^4 \phi}{2l}$ where r is the radius, $\eta$ is the modulus of rigidity and the length l of the whole wire $\phi$. Thus

For equilibrium $WR = \frac{\pi \eta r^4 \phi}{2l}$ — (1)

If C is be the tortional rigidity of the wire the potential energy of the wire will be $\frac{1}{2} C\phi^2$

where $C = \frac{\pi \eta r^4}{2l}$

$\therefore$ Strained P.E. $= \frac{\pi \eta r^4 \phi}{4l}$

Let us suppose that the load W is

depressed and then released. If the downward displacement is $x$, the twist of the wire at the free is increased by $d\phi$ where $x = R\,d\phi$ and the additional work required to produce this displacement is

$$\frac{\pi \eta r^4 (d\phi)^2}{4l} = \frac{\pi \eta r^4}{4l}\left(\frac{x^2}{R^2}\right) \quad \because d\phi = \frac{x}{R}$$

which is additional stained P.E. acquired by the spring. In this position W has been lowered by $x$ and C.G is of the spring by $\frac{x}{2}$ so that the change in P.E. from these effects is

$$\frac{\pi \eta r^4 x^2}{4lR} - Wx - \frac{W_1 x}{2}$$

where $W_1$ is the wt. of spring. At this position K.E. possessed by the load W is $\frac{1}{2}\frac{W}{g}\left(\frac{dx}{dt}\right)^2$ spring itself possesses K.E. which has to be calculated.

Let us consider a element ds at a distance s from fixed end. Hence the instantaneous displacement of the element is $sx/l$ and instantaneous velocity is

$$\frac{d}{dt}\left(\frac{sx}{l}\right) = \frac{s}{l}\left(\frac{dx}{dt}\right)$$

If m be the mass per unit length of K.E. of an element will be

$$\frac{1}{2} ms\left[\frac{s}{l}\cdot\left(\frac{dx}{dt}\right)\right]^2$$

Hence the k.E. of the whole spring is

$$= \frac{1}{2}\frac{m}{l^2}\left(\frac{dx}{dt}\right)^2\int_0^l s^2\,ds = \frac{m}{2l^2}\left(\frac{dx}{dt}\right)^2\cdot\frac{l^3}{3} = \frac{1}{2}\cdot\frac{W_1}{3g}\left(\frac{dx}{dt}\right)^2$$

$\therefore$ Total k.E. $= \frac{1}{2g}\cdot\frac{W}{g}\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}\frac{W_1}{3g}\left(\frac{dx}{dt}\right)^2$

$$= \frac{1}{2g}\left(W + \frac{W_1}{3g}\right)\left(\frac{dx}{dt}\right)^2 \quad \text{— (ii)}$$

From the principle of conservation of energy the total energy of the system should be constant.

$$\therefore \frac{1}{2g}\left(W + \frac{W_1}{3}\right)\left(\frac{dx}{dt}\right)^2 + \frac{\pi\eta r^4 x^2}{4lR^2} - Wx - \frac{W_1 x}{2} = \text{Constant}$$

Diff. w.r.to t, we get,

$$\frac{1}{g}\left(W + \frac{W_1}{3}\right)\left(\frac{d^2x}{dt^2}\right) + \frac{\pi\eta r^4}{4lR^2}\left[x - \frac{2\left(W + \frac{W_1}{2}\right)lR^2}{\pi\eta r^4}\right] = 0 \quad \text{—(iii)}$$

Putting $y = x - \frac{2\left(W + \frac{W_1}{2}\right)lR^2}{\pi\eta r^4}$

We get from eqn. (iii)

$$\frac{1}{g}\left(W + \frac{W_1}{3}\right)\frac{d^2y}{dt^2} + \frac{\pi\eta r^4}{2lR^2}y = 0$$

$$\Rightarrow \frac{d^2y}{dt^2} + \frac{\pi\eta r^4 g}{2lR^2\left(W + \frac{W_1}{3}\right)}y = 0$$

$$\text{or,}\quad \frac{d^2y}{dt^2} + \omega^2 y = 0$$

Where $\omega^2 = \frac{\pi\eta r^4 g}{2lR^2\left(W + \frac{W_1}{3}\right)}$ = a constant — (iv)

Hence from eqn (iv) periodic motion taking place about a displacement zero at the periodic time $T$ given by

$$T = \frac{2\pi}{\omega}$$

$$= 2\pi \sqrt{\frac{2lR^2\left(W + \frac{w_1}{3}\right)}{\pi \eta r^4 g}}$$